Eigen-values and Eigen-vectors

If $\bar{x} \in V \longrightarrow \underbrace{f}_{\text{ENDOM.}}(\bar{x}) = \underbrace{\lambda}_{\in\mathbb{R}} \bar{x}$ $\begin{cases} \bar{x} \text{ is an Eigenvector of } f \text{ for the Eigenvalue } \lambda \\ \lambda \text{ is an Eigenvalue of } f \end{cases}$

Characteristic Equation (C.E.): $\boxed{|F_B - \lambda I| = 0} \longrightarrow \lambda_i$ There are as many Eigenvalues as $\dim(V)$.

$S(\lambda) \equiv$ Eigenspace for Eigenvalue λ $\qquad S(\lambda) = \{ \forall \bar{x} \in V / f(\bar{x}) = \lambda \bar{x} \}$

$$\boxed{S(\lambda) = \text{Ker}(f - \lambda i)}$$

$1 \leqslant \dim(S(\lambda)) \leqslant MO(\lambda)$

$MO(\lambda) \equiv$ Multiplicity Order of Eigenvalue λ $\qquad MO(\lambda) = \text{Max}(\dim(S(\lambda)))$

$MO(\lambda) =$ Multiplicity of λ in the C.E.

e.g.: $|F_B - \lambda I| = 0 \rightarrow \lambda^2(\lambda - 3) = 0 \rightarrow \begin{cases} \lambda_1 = 0 & MO(0) = 2 \\ \lambda_2 = 3 & MO(3) = 1 \end{cases}$

Eigenvalues 1

Given an Endomorph. in $\mathbb{R}^3$, $f(\bar{x}) = (x^1+x^2, x^1+x^2, x^3)$, obtain each and every eigenspace for f with its base and dimension.

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$\begin{cases} \bar{e}_1 = (1,0,0)_B \\ \bar{e}_2 = (0,1,0)_B \\ \bar{e}_3 = (0,0,1)_B \end{cases}$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3)$

C.E.:

$$|F_B - \lambda I| = 0 \longrightarrow \left| \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\longrightarrow \underbrace{\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0}_{|F_B - \lambda I| = 0}$$

$$(1-\lambda)\underbrace{\left[(1-\lambda)^2 - 1\right]}_{\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}} = 0 \rightarrow (1-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$(\lambda-2)(1-\lambda)\lambda = 0 \rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{cases}$$

$MO(0) = 1 \rightarrow 1 \leq \dim(S(0)) \leq \cancel{MO(0)}\ 1 \longrightarrow \dim(S(0)) = 1$

$MO(1) = 1 \rightarrow 1 \leq \dim(S(1)) \leq \cancel{MO(1)}\ 1 \longrightarrow \dim(S(1)) = 1$

$MO(2) = 1 \rightarrow 1 \leq \dim(S(2)) \leq \cancel{MO(2)}\ 1 \longrightarrow \dim(S(2)) = 1$

$$\boxed{S(\lambda)=Ker(f-\lambda i)}$$

$S(0)=Ker(f-0i)=Ker(f)$

$$\underbrace{\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}}_{F_B}\underbrace{\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}}_{\bar{x}}=\underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{\bar{0}}$$

$$\begin{matrix}x^1+x^2=0\\ \cancel{x^1+x^2=0}\\ x^3=0\end{matrix}\qquad\begin{cases}x^1=\gamma\\x^2=-\gamma\\x^3=0\end{cases}\forall\gamma\in\mathbb{R}$$

$S(0)=Ker(f)=\mathcal{L}\{\underbrace{(1,-1,0)}_{\bar{u}_1}\}$

$S(1)=Ker(f-i)$

$$\underbrace{\begin{pmatrix}1-1&1&0\\1&1-1&0\\0&0&1-1\end{pmatrix}}_{F-I}\underbrace{\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}}_{\bar{x}}=\underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{\bar{0}}\longrightarrow\underbrace{\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}}_{F-I}\underbrace{\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}}_{\bar{x}}=\underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{\bar{0}}\longrightarrow\begin{cases}x^2=0\\x^1=0\\ \cancel{0=0}\end{cases}\qquad\begin{cases}x^1=0\\x^2=0\\x^3=\alpha\end{cases}\forall\alpha\in\mathbb{R}$$

$S(1)=\mathcal{L}\{\underbrace{(0,0,1)}_{\bar{u}_2}\}\quad B_{S(1)}=\{(0,0,1)\}$

$S(2)=Ker(f-2i)$

$$\underbrace{\begin{pmatrix}-1&1&0\\1&-1&0\\0&0&-1\end{pmatrix}}_{F-2I}\underbrace{\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}}_{\bar{x}}=\underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{\bar{0}}\longrightarrow\begin{cases}-x^1+x^2=0\to x^1=x^2\\ \cancel{x^1-x^2=0}\\ -x^3=0\longrightarrow x^3=0\end{cases}\qquad\begin{cases}x^1=\beta\\x^2=\beta\\x^3=0\end{cases}\forall\beta\in\mathbb{R}$$

$S(2)=\mathcal{L}\{\overbrace{(1,1,0)}^{\bar{u}_3}\}$

$$\begin{cases} \bar{u}_1 = (1,-1,0)_B \in S(0) \\ \bar{u}_2 = (0,0,1)_B \in S(1) \\ \bar{u}_3 = (1,1,0)_B \in S(2) \end{cases}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0 \qquad B' = \{ \bar{u}_1, \bar{u}_2, \bar{u}_3 \}$$

They are L.I.

$$\begin{cases} \bar{u}_1 = (1,0,0)_{B'} \\ \bar{u}_2 = (0,1,0)_{B'} \\ \bar{u}_3 = (0,0,1)_{B'} \end{cases}$$

B' is a base of Eigen Vectors

$F_{B'}$ is a DIAGONAL matrix with

Eigenvalues in its MAIN DIAGONAL

$$f(\bar{u}_1) = 0 \cdot \bar{u}_1 = \bar{0}$$

$$f(\bar{u}_2) = 1 \cdot \bar{u}_2 = \bar{u}_2$$

$$f(\bar{u}_3) = 2 \cdot \bar{u}_3 = 2\bar{u}_3$$

$$F_{B'} = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ \begin{array}{ccc} f(\bar{u}_1) & f(\bar{u}_2) & f(\bar{u}_3) \end{array} \end{array}$$

in B'

$$\begin{tikzcd} B \arrow[r, bend left, "C"] & B' \arrow[l, bend left, "C^{-1}"] \end{tikzcd}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \begin{array}{ccc} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{array} \end{array}$$

in B

$$C^{-1} = \frac{Adj(C)^t}{|C|}$$

$$\boxed{F_{B'} = C^{-1} F_B \, C}$$

$$\begin{cases} \bar{u}_1 = \bar{e}_1 - \bar{e}_2 \\ \bar{u}_2 = \bar{e}_3 \\ \bar{u}_3 = \bar{e}_1 + \bar{e}_2 \end{cases}$$

$$\bar{u}_1 + \bar{u}_3 = 2\bar{e}_1$$

$$\bar{u}_1 - \bar{u}_3 = -2\bar{e}_2$$

$$\begin{cases} \bar{e}_1 = \frac{1}{2}\bar{u}_1 + \frac{1}{2}\bar{u}_3 \\ \bar{e}_2 = -\frac{1}{2}\bar{u}_1 + \frac{1}{2}\bar{u}_3 \\ \bar{e}_3 = \bar{u}_2 \end{cases}$$

$$C^{-1} = \begin{array}{c} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \\ \begin{array}{ccc} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \end{array} \end{array}$$

in B'

When is an ENDOMORPHISM's matrix able to be diagonalized?

When I can create a base of EIGENVECTORS.

If $f: V \to V$ is an ENDOMORPHISM and B' is a BASE of EIGENVECTORS

$$\boxed{\exists B' \iff OM(\lambda_i) = \dim(S(\lambda_i)) \ \forall i}$$

Eigenvalues 2

Is the endomorphism $f(\bar{x}) = (x^1+x^2+x^3, x^1+x^2+x^3, x^1+x^2+x^3)$ diagonizable ?

$1 \leqslant \dim(S(\lambda)) \leqslant MO(\lambda)$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$f(\bar{e}_1) \quad f(\bar{e}_2) \quad f(\bar{e}_3)$

$$\left.\begin{array}{l} f(\bar{e}_1) - f(\bar{e}_2) = 0 \longrightarrow f(\bar{e}_1 - \bar{e}_2) = \bar{0} \longrightarrow \bar{e}_1 - \bar{e}_2 \in Ker(f) = S(0) \\ f(\bar{e}_2) - f(\bar{e}_3) = \bar{0} \longrightarrow f(\bar{e}_2 - \bar{e}_3) = \bar{0} \longrightarrow \bar{e}_2 - \bar{e}_3 \in Ker(f) = S(0) \end{array}\right\} \dim(S(0)) \geqslant 2$$

so $MO(0) \geqslant 2$

$$f(\bar{e}_1) + f(\bar{e}_2) + f(\bar{e}_3) = 3\bar{e}_1 + 3\bar{e}_2 + 3\bar{e}_3 \longrightarrow f(\bar{e}_1 + \bar{e}_2 + \bar{e}_3) = 3(\bar{e}_1 + \bar{e}_2 + \bar{e}_3)$$

$\bar{e}_1 + \bar{e}_2 + \bar{e}_3 \in S(3)$ $\quad \dim(S(3)) \geqslant 1$

so $MO(3) \geqslant 1$

Since the M.O. of each Eigenvalue is the same as the dimension of the Eigenspaces -> It is DIAGONALIZABLE

Or we solve it in the traditional way:

$$|F_B - \lambda I| = 0 \longrightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0 \longrightarrow [\dots] \longrightarrow (\lambda - 3)\lambda^2 = 0$$